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CITATION:

Yoshino, Masafumi. Singular solutions of Nonlinear Fuchsian Equations and Applications to Normal Form Theory (Deformation of differential equations and asymptotic analysis). 数理解析研究所講究録 2002, 1296: 73-79

ISSUE DATE:

2002-12

URL:

<http://hdl.handle.net/2433/42635>

RIGHT:

Singular solutions of Nonlinear Fuchsian Equations and Applications to Normal Form Theory

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Motivation and Examples

Vector fields with an isolated singular point

Let us consider the following vector field with an isolated singular point at the origin

$$(3) \quad \mathcal{X}(x) = \sum_{j=1}^n a_j(x) \frac{\partial}{\partial x_j},$$

where $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ or \mathbb{C}^n , and $a_j(x)$ is smooth in x . Namely we assume

$$(4) \quad \mathcal{X}(0) = 0,$$

and \mathcal{X} does not vanish in some neighborhood of $x = 0$ except for the origin.

Linearization and Homology Equation

We want to linearize $\mathcal{X}(x)$ by a change of variables

$$(5) \quad x = y + v(y), \quad v = O(|y|^2).$$

We write $\mathcal{X}(x)$ in the form

$$(6) \quad \mathcal{X}(x) = x\Lambda \frac{\partial}{\partial x} + R(x) \frac{\partial}{\partial x} \equiv X(x) \frac{\partial}{\partial x},$$

^{††}Supported by Grant-in Aid for Scientific Research (No. 11640183), Ministry of Education, Science and Culture, Japan and Chuo University, Tokyo, Japan.

2000 AMS Mathematics Subject Classification: primary 35G20 secondary 37C10, 37J40.

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$$\frac{\partial}{\partial x} = \left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n} \right),$$

$$(7) \quad X(x) = x\Lambda + R(x),$$

where

$$(8) \quad R(x) = (R_1(x), \dots, R_n(x)), \quad R(x) = O(|x|^2),$$

and Λ is an $n \times n$ constant matrix.

Noting that

$$\begin{aligned} X(x) \frac{\partial}{\partial x} &= X(y + v(y)) \frac{\partial y}{\partial x} \frac{\partial}{\partial y} \\ &= X(y + v(y)) \left(\frac{\partial x}{\partial y} \right)^{-1} \frac{\partial}{\partial y}, \end{aligned}$$

the **linearization condition** can be written in the following form

$$X(y + v)(1 + \partial_y v)^{-1} = y\Lambda.$$

Therefore

$$(9) \quad (y + v)\Lambda + R(y + v) = y\Lambda(1 + \partial_y v) = y\Lambda + y\Lambda\partial_y v.$$

Hence v satisfies the so-called **homology equation**

$$(*) \quad \mathcal{L}v \equiv y\Lambda\partial_y v - v\Lambda = R(y + v(y)), \quad v = (v_1, \dots, v_n).$$

Summing up we obtain

The necessary and sufficient condition for that $()$ has a solution v is that \mathcal{X} is linearized by the change of substitution $x = y + v(y)$.*

Expression of a homology equation

We assume that Λ is in a diagonal matrix, namely

$$(10) \quad \Lambda = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}.$$

Noting that

$$y\Lambda\partial_y = \sum_{k=1}^n \lambda_k y_k \frac{\partial}{\partial y_k}$$

we obtain

$$(11) \quad \mathcal{L}v = \begin{pmatrix} \sum \lambda_k y_k \frac{\partial}{\partial y_k} - \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \sum \lambda_k y_k \frac{\partial}{\partial y_k} - \lambda_n \end{pmatrix} \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}.$$

In the following, for the sake of simplicity we always assume that a homology equation has the above expression.

Non-resonant condition

The **indicial polynomial** of \mathcal{L} is given by

$$(12) \quad \sum_{k=1}^n \lambda_k \zeta_k - \lambda_j, \quad (j = 1, \dots, n).$$

\mathcal{L} is said to be **non-resonant** if

$$(13) \quad \sum_{k=1}^n \lambda_k \alpha_k - \lambda_j \neq 0$$

for $\forall \alpha \in (\alpha_1, \dots, \alpha_n) \in \mathbf{Z}_+^n$, $|\alpha| \geq 2$, and $j = 1, \dots, n$.

If (13) does not hold we say that \mathcal{L} is **resonant**. The set of y^α with α not satisfying (13) for some j is called a **resonance**. We have

Under non-resonant condition there exists a formal power series solution.

Indeed, $\mathcal{L}v = f$ is written in

$$\mathcal{L}\left(\sum_{\alpha} v_{\alpha} y^{\alpha}\right) = \sum_{\alpha} \left(\sum_{k=1}^n \lambda_k \alpha_k - \Lambda\right) v_{\alpha} y^{\alpha} = \sum_{\alpha} f_{\alpha} y^{\alpha}.$$

Because $(\sum_{k=1}^n \lambda_k \alpha_k - \Lambda)$ is invertible \mathcal{L}^{-1} exists. Because $R(x) = O(|x|^2)$ we can determine a formal power series solution by a method of indeterminate coefficients.

Two theorems for the solvability of a homology equation

Poincaré introduced a famous **Poincaré condition**

$$\operatorname{Re} \lambda_j > 0, \quad j = 1, \dots, n$$

and showed the solvability of (*) in a class of analytic functions.

Solvability of (*) in a real domain

Theorem (Sternberg) *Assume the hyperbolic condition*

$$(14) \quad \operatorname{Re} \lambda_k \neq 0, \quad k = 1, \dots, n.$$

Moreover, suppose the non-resonant condition. Then () has a smooth solution.*

If resonance occurs we have

Theorem (Grobman- Hartman) *Assume the hyperbolicity. Then (*) has a continuous solution.*

Remark A continuous solution of (*) is defined as a **weak solution**. The definition of a weak solution is standard. There are extensions of this result to the C^k ($k \geq 0$) case by Blitskiy et. al for a certain class of vector fields with resonances.

Object of Study

We want to solve (*) in the case of resonances in a class of functions with a "log" type singularity. We also want to solve (*) in a class of functions holomorphic in the domain which is a product of sectors with vertex at the origin.

Statement of the results

Singular solutions

Theorem 1. *Assume the Poincaré condition and*

$$\forall i, j, k, \quad \lambda_i + \lambda_j \neq \lambda_k.$$

Then Eq. () has a solution v of the form*

$$v(y) = \sum_{|\alpha| \geq 2, \alpha \geq \beta} v_{\alpha\beta} y^{\alpha} (\log y)^{\beta},$$

where $(\log y)^\beta = \prod_{j=1}^n (\log y_j)^{\beta_j}$. $v(y)$ converges in

$$\{y \in \mathcal{C}^m; |y| < \exists \varepsilon, |y_j \log y_j| < \varepsilon (j = 1, \dots, n)\}.$$

Remark. If there is no resonance the above solution is a classical solution constructed by Poincaré.

If we restrict the solution v to the real domain we obtain a finitely smooth solution of (*). Hence **a finite smoothness occurs because of the log type singularity caused by the resonance.**

Example Consider the case $n = 2$. Let $m \geq 2$ be an integer. Let us consider

$$\mathcal{L}_1 = x_1 \partial_1 + m x_2 \partial_2 - 1, \quad \mathcal{L}_2 = x_1 \partial_1 + m x_2 \partial_2 - m.$$

The only resonance is $(\alpha_1, \alpha_2) = (m, 0)$. The solution v has singularity of $\log x_1$ type.

Indeed, the resonance $\alpha = (\alpha_1, \alpha_2) \in \mathbb{Z}_+^2$ satisfies $\alpha_1 + \alpha_2 \geq 2$ and

$$\alpha_1 + m \alpha_2 - 1 = 0, \quad \text{or} \quad \alpha_1 + m \alpha_2 = m.$$

Since $\alpha_1 + m \alpha_2 - 1 \neq 0$ by assumption we obtain $\alpha_1 + m \alpha_2 = m$ and $\alpha_1 + \alpha_2 \geq 2$. It follows that $(\alpha_1, \alpha_2) = (m, 0)$.

Sketch of the proof of Theorem 1. For the sake of simplicity we will prove the above example. We will construct a formal solution of (*) in the following form

$$u_j(x) = \sum_{\alpha \in \mathbb{Z}_+^2, |\alpha| \geq 2, k} u_{\alpha, k}^j x^\alpha (\log x_1)^k, \quad j = 1, 2.$$

The equation (*) can be written in the following form

$$(*) \quad \mathcal{L}_j u_j = R_j(x_1 + u_1, x_2 + u_2), \quad j = 1, 2.$$

We set $u_{\alpha, k} = (u_{\alpha, k}^1, u_{\alpha, k}^2)$. We determine $u_{\alpha, k}$ $k = 0, 1, 2, \dots$ inductively. We determine $u_{\alpha, 0}$. By comparing the coefficients we can determine $u_{\alpha, 0}$ for $|\alpha| \leq m, \alpha \neq (m, 0)$. On the other hand we note

$$\mathcal{L}_2(x_1^m) = 0, \quad \mathcal{L}_2(x_1^m \log x_1) = x_1^m.$$

Hence we set $u_{(m,0),0}^2 = 0$, $u_{(m,0),0} = (u_{(m,0),0}^1, 0)$. We note that we can determine $u_{(m,0),0}^1$ and $u_{(m,0),1}^2$ by comparing the coefficients of x_1^m in (*) since \mathcal{L}_1 has the nonresonance property. It is clear that we can determine $u_{\alpha,0}$ for $|\alpha| > m$ from (*) because there is no resonance for $|\alpha| > m$.

We next determine $u_{\alpha,1}$. We have already determined $u_{(m,0),1} = (0, u_{(m,0),1}^2)$. By the nonresonance property we can determine $u_{\alpha,1}$ for $|\alpha| > m$. Inductively, $u_{\alpha,2}$ ($|\alpha| = 2m$) can be determined by comparing the coefficients of $x_1^{2m}(\log x_1)^2$. The terms $u_{\alpha,2}$ ($|\alpha| > 2m$) can be determined inductively by the nonresonance property. Inductively, we can determine $u_{\alpha,k}$ ($k = 0, 1, 2, \dots$). Hence we can determine a formal power series solution. The convergence can be proved by the method of majorant series. This ends the proof.

Solvability in the sectorial domain

Let S_0 be a sector in the complex plane, $S_0 := \{z; |\arg z| < \theta\}$, where $\theta > 0$ is a given small number and the branch of $\arg z$ is taken so that the argument is zero on the real axis. We define a sectorial domain S in \mathbb{C}^n as the product of n copies of S_0 , $S = S_0 \times \dots \times S_0$. In the following we consider the solvability of the equation (*) in the sectorial domain S . The typical example of the nonlinear term $R(x)$ is the following:

$$R(x) = A \prod_{j=1}^n \frac{x_j^{\alpha_j}}{(x_j - c_j)^{\beta_j}},$$

where $A, c_j \in \mathbb{C} \setminus \bar{S}$, $0 < \alpha_j < \beta_j$ ($j = 1, \dots, n$) are constants. We set $\lambda := (\lambda_1, \dots, \lambda_n)$. Then we have

Theorem 2. *Suppose that*

$$\lambda_j \in \mathbb{R} \setminus 0 \quad (j = 1, \dots, n).$$

Let $\Gamma \subset \mathbb{R}^n$ be an open set such that $0 \in \Gamma$ and

$$\Gamma \cap \{\eta; \langle \lambda, \eta \rangle = \lambda_j\} = \emptyset,$$

for every $j = 1, \dots, n$, where $\langle \lambda, \eta \rangle = \sum_{k=1}^n \lambda_k \eta_k$. Suppose that, for every $\eta \in \Gamma$,

$$R(x) = O(x^{-\eta}), \quad (\text{when } x \rightarrow 0 \text{ or } x \rightarrow \infty, x \in S).$$

Then there exists $\varepsilon > 0$ such that if $\sup_{x \in S} |R(x)| < \varepsilon$ the equation (*) has a solution u holomorphic in S . Moreover, for every $\eta \in \Gamma$, u behaves like $O(x^{-\eta})$ when $x \rightarrow 0$ or $x \rightarrow \infty$ $x \in S$.

Example. For $R(x)$ in the above example the conditions in the theorem are fulfilled if Γ is a sufficiently small neighborhood of the origin and A is sufficiently small.

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